

DUALITIES OF ARTINIAN COALGEBRAS WITH APPLICATIONS TO NOETHERIAN COMPLETE ALGEBRAS

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ABSTRACT. A duality theorem of the bounded derived category of quasi-finite comodules over an artinian coalgebra is established. Let A be a noetherian complete basic semiperfect algebra over an algebraically closed field, and C be its dual coalgebra. If A is Artin-Schelter regular, then the local cohomology of A is isomorphic to a shift of twisted bimodule ${}_1C_{\sigma^*}$ with σ a coalgebra automorphism. This yields that the balanced dualizing complex of A is a shift of the twisted bimodule ${}_{\sigma^*}A_1$. If σ is an inner automorphism, then A is Calabi-Yau.

INTRODUCTION

The noncommutative dualizing complex, introduced by Yekutieli in [23], provides a powerful tool to study noncommutative algebras. In order to determine a dualizing complex over a graded algebra, Yekutieli introduced the concept of a *balanced* dualizing complex. However, the term ‘balanced’ makes no sense for a general non-graded algebra. As an alternative, Van den Bergh introduced the concept of a *rigid* dualizing complex (cf. [21]) for a non-graded algebra. The existence of a rigid (or balanced) dualizing complex of an algebra is related to the (twisted) Calabi-Yau property of certain triangulated category (cf. [11]). Van den Bergh’s results were generalized to noetherian complete semilocal algebras (cf. [9, 22]). Many good properties of noetherian complete semilocal algebras were discovered through dualizing complexes.

However, any noetherian complete algebra A with cofinite Jacobson radical is the dual algebra of an artinian coalgebra C (cf. [15]). There exist certain duality properties between the category of A -modules and the category of C -comodules. This motivates us to study the balanced dualizing complex, the Calabi-Yau property and the local cohomology of a noetherian complete algebra through artinian coalgebras. To this aim, we first have to discuss some homological properties of artinian coalgebras.

Let C be an artinian coalgebra. In Section 1, we establish some dualities between triangulated subcategories of derived categories of C^* -modules and those of C -comodules. In particular, it turns out that the dual algebra C^* is Calabi-Yau if and only if C is.

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Through the dualities obtained in Section 1, we deduce the following duality theorem of the bounded derived categories of left and of right C -comodules when C satisfies certain additional conditions in Section 2.

Theorem. *Let C be an artinian coalgebra. If the following conditions are satisfied:*

- (i) *the functors $\Gamma : {}_C {}^* \mathcal{M} \longrightarrow \mathcal{M}^C$ and $\Gamma^\circ : \mathcal{M}_{C^*} \longrightarrow {}^C \mathcal{M}$ have finite cohomological dimensions;*
- (ii) *the coalgebra C satisfies the left and the right χ -condition,*

then the functors $F = R\Gamma \circ ()^$ and $G = R\Gamma^\circ \circ ()^*$ are dualities of triangulated categories:*

$$D_{qf}^b({}^C \mathcal{M}) \xrightleftharpoons[G]{F} D_{qf}^b(\mathcal{M}^C).$$

In the theorem, ${}_C {}^* \mathcal{M}$ is the category of left C^* -modules, ${}^C \mathcal{M}$ is the category of left C -comodules, and $D_{qf}^b({}^C \mathcal{M})$ is the derived category of bounded complexes of left C -comodules with quasi-finite cohomology comodules. The χ -condition in the theorem will be explained in the section 2, and $\Gamma = \text{Rat}$ is the rational functor.

In Section 3 and Section 4, we focus on a class of artinian coalgebras satisfying the χ -condition, namely the Artin-Schelter regular coalgebras. Let A be a noetherian complete basic algebra over an algebraically closed field, and let $C = A^\circ$ be its dual coalgebra. If A has cofinite Jacobson radical, then $A \cong C^*$. If A is Artin-Schelter regular, then C is an Artin-Schelter regular coalgebra. We have the following theorem (cf. Theorem 3.7 and Corollary 4.3).

Theorem. *Let A be a noetherian complete basic algebra with cofinite Jacobson radical over an algebraically closed field, and C be its dual coalgebra. Assume that A is Artin-Schelter regular of global dimension n . Then*

- (i) *there is a coalgebra automorphism $\sigma \in \text{Aut}(C)$ such that $R\Gamma(A) \cong {}_1 C_{\sigma^*}[-n]$ in $\mathcal{D}^b({}_A \mathcal{M}_A)$;*
- (ii) *for any finitely generated left (or right) A -module, $\dim \text{Ext}_A^n(M, A) < \infty$; moreover, as vector spaces $\text{Ext}_A^n(M, A) \cong \text{Rat}(M)^*$;*
- (iii) *for $i < n$, $\text{Ext}_A^i(M, A) \cong \text{Ext}_A^i(M/\text{Rat}(M), A)$.*
- (iv) *if the automorphism σ in (i) is inner, then A is Calabi-Yau.*

The items (ii) and (iii) can be viewed as generalizations of [2, Prop. 2.46(ii,iii)] and [25, Theorem 0.3(4)]. The item (i) says that A has a balanced dualizing complex ${}_\alpha A_1[n]$ (α is an algebra automorphism), which is similar to connected graded Artin-Schelter regular algebras (cf. [23]).

Throughout \mathbf{k} is an algebraically closed field of characteristic zero. All the algebras and coalgebras involved are over \mathbf{k} ; unadorned \otimes means $\otimes_{\mathbf{k}}$ and Hom means $\text{Hom}_{\mathbf{k}}$. Let C be a coalgebra, and M and N be left (or right) C -comodules. We use $\text{Hom}_C(M, N)$ to denote the set of left (or right) C -comodule morphisms. To avoid possible confusion, sometimes we use $\text{Hom}_{C^{op}}(M, N)$ to denote the set of right C -comodule morphisms for two C -bicomodules M and N . We use similar Notations for modules over an algebra.

1. DUALITIES BETWEEN AN ARTINIAN COALGEBRA AND ITS DUAL ALGEBRA

Let C be an artinian coalgebra. Let $\text{Rat}(\mathcal{M})_{C^*}$ be the subcategory of \mathcal{M}_{C^*} consisting of rational modules. Then the abelian category $\text{Rat}(\mathcal{M})_{C^*}$ is equivalent to the abelian category ${}^C\mathcal{M}$.

For a coalgebra C , ${}^C C$ is an injective object in ${}^C\mathcal{M}$, or equivalently, C_{C^*} is an injective object in $\text{Rat}(\mathcal{M})_{C^*}$. In general, C_{C^*} is not injective in \mathcal{M}_{C^*} . However, we have the following property (see [5, 9.4] and [7, Theorem 3.2]).

Proposition 1.1. *The following are equivalent.*

- (i) C_{C^*} is injective in \mathcal{M}_{C^*} ;
- (ii) C_{C^*} is an injective cogenerator of \mathcal{M}_{C^*} ;
- (iii) ${}_{C^*}C$ is artinian;
- (iv) C^* is left noetherian;
- (v) The injective hull of a rational left C^* -module is rational.

From now on, unless stated otherwise, C is both a left and a right artinian coalgebra.

Let R and S be two rings, ${}_R E_S$ an R - S -bimodule. Recall from [1] that an R -module ${}_R M$ (or an S -module N_S) is called an E -reflexive if the natural morphism

$$M \longrightarrow \text{Hom}_S(\text{Hom}_R(M, E), E) \quad (\text{or, } N \longrightarrow \text{Hom}_R(\text{Hom}_S(N, E), E))$$

is an isomorphism. An R - S -bimodule ${}_R E_S$ defines a *Morita duality* [1, Sect. 24] if

- (i) both ${}_R R$ and S_S are E -reflexive;
- (ii) every submodule and every quotient module of an E -reflexive module is E -reflexive.

For a coalgebra C , consider the C^* -bimodule ${}_{C^*}C_{C^*}$. Clearly we have $\text{End}(C_{C^*}) \cong C^*$ and $\text{End}({}_{C^*}C) \cong C^{*op}$. Moreover, if C is (both left and right) artinian, then both ${}_{C^*}C$ and C_{C^*} are injective cogenerators by Prop. 1.1. So, the C^* -bimodule ${}_{C^*}C_{C^*}$ defines a Morita duality [1, Theorem 24.1]. Let \mathcal{U}_{C^*} (resp. ${}_{C^*}\mathcal{U}$) be the subcategory

of \mathcal{M}_{C^*} (resp. ${}_{C^*}\mathcal{M}$) consisting of ${}_{C^*}C_{C^*}$ -reflexive modules. Then we have a duality:

$$(1) \quad \mathcal{U}_{C^*} \xrightleftharpoons[\text{Hom}_{C^*}(-, C)]{\text{Hom}_{C^*}(-, C)} {}_{C^*}\mathcal{U}.$$

Let ${}_{C^*}\mathcal{M}_{fg}$ be the category of all finitely generated left C^* -modules, and let ${}^C\mathcal{M}_{qf}$ be the category of all quasi-finite left C -comodule. From the duality (1), we immediately obtain the following.

Proposition 1.2. *Let C be an artinian coalgebra. There is a duality between abelian categories:*

$${}_{C^*}\mathcal{M}_{fg} \xrightleftharpoons[(\)^*]{\text{Hom}_{C^*}(-, C)} {}^C\mathcal{M}_{qf}.$$

Since the dual algebra C^* of an artinian coalgebra is noetherian and semiperfect, the proposition above immediately follows the next corollary (see also [19, Prop. 3.6] and [7, Prop. 3.4]).

Corollary 1.3. *If C is an artinian coalgebra, then $gl.dim C = gl.dim(C^*)$.*

Let C be an arbitrary coalgebra, ${}_{C^*}M$ a C^* -module. There is a right C^* -module morphism:

$$\varphi_M : \text{Hom}_{C^*}(M, C) \rightarrow M^*, \quad f \mapsto \varepsilon \circ f.$$

If M is a finitely generated C^* -module, then the right C^* -module $\text{Hom}_{C^*}(M, C)$ is in fact a rational C^* -module. Moreover, if we view $\text{Hom}_{C^*}(M, C)$ as a left C -comodule, then it is a quasi-finite comodule. Hence the image of φ_M is contained in $\text{Rat}(M^*)$. Therefore, we obtain a natural transformation:

$$(2) \quad \varphi : \text{Hom}_{C^*}(-, C) \longrightarrow \text{Rat} \circ (\)^*,$$

of functors from ${}_{C^*}\mathcal{M}_{fg}$ to ${}^C\mathcal{M}_{qf}$.

Lemma 1.4. *If C is an artinian coalgebra, then the natural transformation φ above is a natural isomorphism.*

Proof. If ${}_{C^*}M$ is a finitely generated free module, then $\varphi_M : \text{Hom}_{C^*}(M, C) \longrightarrow \text{Rat} \circ (M)^*$ is clearly a isomorphism. For a general finitely generated module M , M is finitely presented since C^* is noetherian:

$$\bigoplus_{\text{finite}} C^* \longrightarrow \bigoplus_{\text{finite}} C^* \longrightarrow M \longrightarrow 0.$$

The statement follows from the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{C^*}(M, C) & \longrightarrow & \mathrm{Hom}_{C^*}\left(\bigoplus_{\text{finite}} C^*, C\right) & \longrightarrow & \mathrm{Hom}_{C^*}\left(\bigoplus_{\text{finite}} C^*, C\right) \\
& & \downarrow \varphi_M & & \downarrow \varphi & & \downarrow \varphi \\
0 & \longrightarrow & \mathrm{Rat} \circ (M)^* & \longrightarrow & \mathrm{Rat} \circ \left(\bigoplus_{\text{finite}} C^*\right)^* & \longrightarrow & \mathrm{Rat} \circ \left(\bigoplus_{\text{finite}} C^*\right)^* . \quad \square
\end{array}$$

Let A be a noetherian algebra with Jacobson radical J such that $A_0 = A/J$ is finite dimensional. Let ${}_A M$ be an A -module. An element $m \in M$ is called a *torsion element* if $J^n m = 0$ for $n \gg 0$. Let $\Gamma(M) = \{m \in M \mid m \text{ is a torsion element}\}$. Then $\Gamma(M)$ is a submodule of M . In fact, we have an additive functor [22]

$$\Gamma : {}_A \mathcal{M} \longrightarrow {}_A \mathcal{M},$$

by sending an A -module to its maximal torsion submodule. Clearly, Γ is a left exact functor. The functor Γ has another representation $\Gamma(M) = \varinjlim \mathrm{Hom}_A(A/J^n, M)$. We use Γ° to denote the torsion functor on the category of right A -modules.

Now let C be a coalgebra, and let

$$\mathrm{Rat} : {}_{C^*} \mathcal{M} \longrightarrow {}_{C^*} \mathcal{M}$$

be the rational functor. If C is artinian, then by [15, Prop. 3.1.1 and Remarks 3.1.2] every finite dimensional C^* -module is rational. Hence, for a left C^* -module M , we have that $\mathrm{Rat}(M)$ is the sum of all the finite dimensional submodules of M . On the other hand, since the Jacobson radical $J = C_0^\perp$ and C_0 is finite dimensional, $\Gamma(M)$ is also the sum of all the finite dimensional submodules of M . Hence $\Gamma(M) \cong \mathrm{Rat}(M)$. So, the functor Γ is naturally isomorphic to the rational functor Rat . In what follow, we identify the right derived functor $R\Gamma$ with $R\mathrm{Rat}$.

Let C be an artinian coalgebra. Then $\mathrm{Soc}(C)$ is finite dimensional. This means that there are only finitely many non-isomorphic simple right (or left) C -comodules. If ${}^C M$ is quasi-finite then $\mathrm{Soc}(M)$ is finite dimensional. Thus ${}^C M$ is finitely cogenerated. This implies that ${}^C \mathcal{M}_{qf}$ is a thick subcategory of ${}^C \mathcal{M}$. Hence $\mathcal{D}_{qf}^+({}^C \mathcal{M})$, the derived category of bounded below complexes of left C -comodule with quasi-finite cohomology comodules, is a full triangulated subcategory of $\mathcal{D}^+({}^C \mathcal{M})$. Also, since C^* is noetherian, $\mathcal{D}_{fg}^-({}_{C^*} \mathcal{M})$, the derived category of bounded above complexes of left C^* -modules with finitely generated cohomology modules, is a full triangulated subcategory of $\mathcal{D}^-({}_{C^*} \mathcal{M})$. The duality in Prop. 1.2 induces a duality of derived categories.

Proposition 1.5. *Let C be an artinian coalgebra. We have dualities of triangulated categories:*

$$\mathcal{D}_{fg}^-(C^*\mathcal{M}) \xrightleftharpoons[(\)^*]{R\Gamma^\circ \circ (\)^*} \mathcal{D}_{qf}^+({}^C\mathcal{M}), \quad \mathcal{D}_{fg}^b(C^*\mathcal{M}) \xrightleftharpoons[(\)^*]{R\Gamma^\circ \circ (\)^*} \mathcal{D}_{qf}^b({}^C\mathcal{M}).$$

Proof. Since C is artinian, $\mathcal{D}_{qf}^+({}^C\mathcal{M})$ is equivalent to $\mathcal{D}^+({}^C\mathcal{M}_{qf})$, the derived category of complexes of quasi-finite comodules. In the dual case, C^* is noetherian, and $\mathcal{D}_{fg}^-(C^*\mathcal{M})$ is equivalent to $\mathcal{D}^-(C^*\mathcal{M}_{fg})$. By Prop. 1.2, we have the following duality

$$\mathcal{D}^-(C^*\mathcal{M}_{fg}) \xrightleftharpoons[(\)^*]{\text{Hom}_{C^*}(-, C)} \mathcal{D}^+({}^C\mathcal{M}_{qf}).$$

Now using Lemma 1.4 one may check without difficulty that the composition

$$\mathcal{D}_{fg}^-(C^*\mathcal{M}) \xrightarrow{\cong} \mathcal{D}^-(C^*\mathcal{M}_{fg}) \xrightarrow{\text{Hom}_{C^*}(-, C)} \mathcal{D}^+({}^C\mathcal{M}_{qf}) \xrightarrow{\cong} \mathcal{D}_{qf}^+({}^C\mathcal{M})$$

is naturally isomorphic to the functor $R\Gamma^\circ \circ (\)^*$. Moreover, one sees that $R\Gamma^\circ \circ (\)^*$ sends bounded complexes to bounded complexes. \square

Corollary 1.6. *Let C be an artinian coalgebra. Then we have a duality of triangulated categories:*

$$\mathcal{D}_{fd}^b(C^*\mathcal{M}) \xrightleftharpoons[(\)^*]{R\Gamma^\circ \circ (\)^*} \mathcal{D}_{fd}^b({}^C\mathcal{M}).$$

Proof. It suffices to show that $R\Gamma^\circ(M)^* \in \mathcal{D}_{fd}^b({}^C\mathcal{M})$ for all finite dimensional left C^* -module M . Since C^* is noetherian and complete with respect to the radical filtration, the Jacobson radical J of C^* satisfies Artin-Rees condition. Hence the injective envelop of a J -torsion module is still J -torsion (cf. [7, Theorem 3.2]). Now M^* is a J -torsion module. We have an injective resolution of M^* with each component being J -torsion. Hence $R\Gamma(M^*)$ is quasi-isomorphic to M^* , that is, $R\Gamma(M^*) \in \mathcal{D}_{fd}^b({}^C\mathcal{M})$. \square

Recall that a \mathbf{k} -linear category \mathcal{C} is said to be *Hom-finite*, if for any $X, Y \in \mathcal{C}$, $\text{Hom}_{\mathcal{T}}(X, Y)$ is a finite dimensional \mathbf{k} -vector space; a Hom-finite \mathbf{k} -linear triangulated category \mathcal{T} is called a *Calabi-Yau category of dimension n* if, for any objects $X, Y \in \mathcal{T}$, there is a natural isomorphism $\text{Hom}_{\mathcal{T}}(X, Y) \cong \text{Hom}_{\mathcal{T}}(Y, X[n])^*$; an algebra A is called a (left) *Calabi-Yau of dimension n* (simply, CY- n) if

- (i) $\mathcal{D}_{fd}^b({}_A\mathcal{M})$ is Hom-finite;
- (ii) $\mathcal{D}_{fd}^b({}_A\mathcal{M})$ is a Calabi-Yau category of dimension n .

Note that the CY property of an algebra is always left-right symmetric. Thus we simply say that an algebra is CY- n .

In the dual case, we say that a coalgebra is (left) CY- n if

- (i) $\mathcal{D}_{fd}^b({}^C\mathcal{M})$ is Hom-finite;
- (ii) $\mathcal{D}_{fd}^b({}^C\mathcal{M})$ is a Calabi-Yau category of dimension n .

In general, we don't know whether the CY property of a coalgebra is left-right symmetric. But an artinian coalgebra is left CY if and only if it is right CY. In fact, from the definitions we have the following.

Corollary 1.7. *Let C be an artinian coalgebra. Then C is (left) CY- n if and only if C^* is (left) CY- n .*

Example 1.8. Let C be the path coalgebra [8] of the quiver Q with one vertex and one arrow. Then the dual algebra C^* is the formal power series algebra $\mathbf{k}[[x]]$. It is well known that $\mathbf{k}[[x]]$ is a CY-1 algebra. Hence C is CY-1 coalgebra.

2. DUALITIES OF COMODULES OVER AN ARTINIAN COALGEBRA

In this section, we establish a duality of the derived categories of left C -comodules and of right C -comodules by using the results obtained in Section 1.

As \mathcal{M}_{qf} is a thick subcategory of ${}^C\mathcal{M}$, $D_{qf}^b({}^C\mathcal{M})$ is a full triangulated subcategory of $D^b({}^C\mathcal{M})$. Consider the following functors:

$$D^b({}^C\mathcal{M}) \xrightarrow{(\)^*} D^b({}_{C^*}\mathcal{M}) \xrightarrow{R\Gamma} D^+(\mathcal{M}^C).$$

We want to know when the composite $R\Gamma \circ (\)^*$ has its image in $D^b(\mathcal{M}^C)$. If this happens, does the restriction of $R\Gamma \circ (\)^*$ to the subcategory $D_{qf}^b({}^C\mathcal{M})$ result a functor $D_{qf}^b({}^C\mathcal{M}) \rightarrow D_{qf}^b(\mathcal{M}^C)$?

We need to translate some concepts relative to noncommutative algebras to coalgebras.

Definition 2.1. Let C be an artinian coalgebra. We say that a quasi-finite left C -comodule M satisfies the χ -condition if, for every simple left C -comodule S , $\text{Ext}_C^i(M, S)$ is finite dimensional for all $i \geq 0$. We say that a coalgebra C satisfies the *left χ -condition* if every quasi-finite left C -comodule satisfies the χ -condition. Similarly, we can define the *right χ -condition*.

The χ -condition on a coalgebra is dual to a similar condition on a noetherian algebra, which is originally introduced in [3] for noetherian graded algebras and was extended to nongraded algebras in [22, 9].

Definition 2.2. Let A be a noetherian algebra with Jacobson radical J such that $A_0 = A/J$ is finite dimensional. A finitely generated left A -module M is said to satisfy the χ -condition if $\text{Ext}_A^i(A_0, M)$ is finite dimensional for all $i \geq 0$. A is said to satisfy the *left χ -condition* if every finitely generated left A -module satisfies the χ -condition.

Let ${}_{C^*}N$ be a module. There is a natural C^* -module morphism

$$\eta_N : N \rightarrow \Gamma(N^*)^*, \quad \eta(n)(f) = f(n)$$

for $n \in N$ and $f \in \Gamma(N^*)$. Observe that if N is a finitely generated free C^* -module, then η_N is an isomorphism.

Lemma 2.3. *If ${}_{C^*}N$ is a finitely generated module, then $R\Gamma(N^*)$ is quasi-isomorphic to $\Gamma(N^*)$.*

Proof. Choose a projective resolution of N as follows:

$$0 \longleftarrow N \longleftarrow P^0 \longleftarrow P^{-1} \longleftarrow \cdots \longleftarrow P^{-i} \longleftarrow \cdots,$$

where each P^{-i} is a finitely generated free C^* -module. Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longleftarrow & N & \longleftarrow & P^0 & \longleftarrow & \cdots \longleftarrow P^i \longleftarrow \cdots \\ & & \downarrow \eta_N & & \downarrow \eta_{P^0} & & \downarrow \eta_{P^i} \\ 0 & \longleftarrow & \Gamma(N^*)^* & \longleftarrow & \Gamma(P^0)^* & \longleftarrow & \cdots \longleftarrow \Gamma(P^{-i})^* \longleftarrow \cdots \end{array}$$

Since P^{-i} is a finitely generated free C^* -module, $\eta_{P^{-i}}$ is an isomorphism for all $i \geq 0$. Hence η_N is an isomorphism since Γ is left exact. Now exactness of the top row implies that the bottom row is exact too. Then the sequence

$$0 \longrightarrow \Gamma(N^*) \longrightarrow \Gamma(P^0)^* \longrightarrow \cdots \longrightarrow \Gamma(P^{-i})^* \longrightarrow \cdots$$

is also exact. Note that the sequence $P^{\bullet*}$ is an injective resolution of N^* . Hence $R\Gamma(N^*)$ is quasi-isomorphic to $\Gamma(N^*)$. \square

Proposition 2.4. *C satisfies the left χ -condition if and only if C^* satisfies the left χ -condition.*

Proof. Note that the left χ -condition on C is equivalent to the condition that for any quasi-finite left C -comodule M and any finite dimensional comodule K , $\text{Ext}_C^i(M, K)$ is finite dimensional for all $i \geq 0$. Let N be any finitely generated C^* -module and L be a finite dimensional C^* -module. By Prop. 1.5, $\text{Ext}_{C^*}^i(L, N) = \text{Hom}_{\mathcal{D}_{fg}^b(C^*\mathcal{M})}(L, N[i]) \cong \text{Hom}_{\mathcal{D}_{qf}^b(C\mathcal{M})}(R\Gamma(N^*), R\Gamma(L^*)[i])$. Since $R\Gamma(N^*)$ and $R\Gamma(L^*)$ are quasi-isomorphic with N^* and L^* respectively (see the proof of Corollary 1.6), we have isomorphism: $R\Gamma(L^*)$ is quasi-isomorphic to L^* , $\text{Ext}_{C^*}^i(L, N) \cong \text{Hom}_{\mathcal{D}_{qf}^b(C\mathcal{M})}(\Gamma(N^*), L^*[i])$. Now if C satisfies the left χ -condition, $\text{Hom}_{\mathcal{D}_{qf}^b(C\mathcal{M})}(\Gamma(N^*), L^*[i])$ is finite dimensional for all $i \geq 0$. It follows that $\text{Ext}_{C^*}^i(L, N)$ is finite dimensional for all $i \geq 0$. Similarly we see the converse is also true. \square

Now we may establish a Morita-type duality of the derived categories of comodules. A Morita-type equivalence of the derived categories of comodules was shown in [10].

Theorem 2.5. *Let C be an artinian coalgebra. If the following conditions are satisfied:*

- (i) the functors $\Gamma : {}_C\mathcal{M} \longrightarrow \mathcal{M}^C$ and $\Gamma^\circ : \mathcal{M}_{C^*} \longrightarrow {}^C\mathcal{M}$ have finite cohomological dimensions;
- (ii) the coalgebra C satisfies the left and the right χ -conditions,

then the functors $F = R\Gamma \circ ()^*$ and $G = R\Gamma^\circ \circ ()^*$ are dualities of triangulated categories:

$$D_{qf}^b({}^C\mathcal{M}) \xrightleftharpoons[G]{F} D_{qf}^b(\mathcal{M}^C).$$

Proof. First of all, we have to show that the functors F and G are well-defined. For an object $X \in D_{qf}^b({}^C\mathcal{M})$, X^* lies in $D_{fg}^b({}_{C^*}\mathcal{M})$. By (i), Γ has finite cohomological dimension. Then $R\Gamma(X^*)$ is in $D^b(\mathcal{M}^C)$. We show that $R\Gamma(X^*)$, in fact, belongs to $D_{qf}^b(\mathcal{M}^C)$. Since $X \in D_{qf}^b({}^C\mathcal{M})$, there is no harm to assume that X is a finitely cogenerated C -comodule. Now X^* is a finitely generated C^* -module. Let $I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots \longrightarrow I^i \longrightarrow \dots$ be a minimal injective resolution of X^* . By (ii), C satisfies the left χ -condition. Then $\text{Hom}_{C^*}(S, I^i)$ is finite dimensional for all simple left C^* -module S and all $i \geq 0$. Hence $\text{Soc}(I^i)$ is finite dimensional for all $i \geq 0$. Note that $\text{Soc}(I^i) \subseteq \text{Rat}(I^i) = \Gamma(I^i)$ and $\text{Soc}(I^i)$ is essential in $\Gamma(I^i)$. We obtain that $\Gamma(I^i)$ is finitely cogenerated as a right C -module for all $i \geq 0$. Then $R\Gamma(X^*)$, which is quasi-isomorphic to $\Gamma(I^\bullet)$, must have finitely cogenerated C -comodules as its cohomologies. Hence $F(X) = R\Gamma(X^*) \in D_{qf}^b(\mathcal{M}^C)$. Similarly, we can show that $G(Y)$ lies in $D_{qf}^b({}^C\mathcal{M})$ for any $Y \in D_{qf}^b(\mathcal{M}^C)$.

By assumption, C^* is noetherian, complete and semiperfect. Thus C^*/J is semisimple and finite dimensional. Applying Theorem 4.1 of [22] to our case, we obtain a duality of the following triangulated categories:

$$\mathcal{D}_{fg}^b(\mathcal{M}_{C^*}) \xrightleftharpoons[\text{Hom}_{C^*}(R\Gamma(-), C)]{\text{Hom}_{C^*}(R\Gamma^\circ(-), C)} \mathcal{D}_{fg}^b({}_{C^*}\mathcal{M}).$$

On the other hand, following Prop. 1.5, we have dualities

$$\mathcal{D}_{fg}^b({}_{C^*}\mathcal{M}) \xrightleftharpoons[()^*]{R\Gamma \circ ()^*} \mathcal{D}_{qf}^b({}^C\mathcal{M}), \quad \mathcal{D}_{fg}^b(\mathcal{M}_{C^*}) \xrightleftharpoons[()^*]{R\Gamma^\circ \circ ()^*} \mathcal{D}_{qf}^b(\mathcal{M}^C).$$

We want to show that the composition

$$\Psi : \mathcal{D}_{qf}^b({}^C\mathcal{M}) \xrightarrow{()^*} \mathcal{D}_{fg}^b({}_{C^*}\mathcal{M}) \xrightarrow{\text{Hom}_{C^*}(R\Gamma(-), C)} \mathcal{D}_{fg}^b(\mathcal{M}_{C^*}) \xrightarrow{R\Gamma^\circ \circ ()^*} \mathcal{D}_{qf}^b(\mathcal{M}^C)$$

is naturally isomorphic to the functor $F = R\Gamma \circ ()^*$. For any $X \in \mathcal{D}_{qf}^b({}^C\mathcal{M})$, from the last paragraph we see $R\Gamma(X^*) \in \mathcal{D}_{qf}^b(\mathcal{M}^C)$. We have natural isomorphisms

$$\text{Hom}_{C^*}(R\Gamma(X^*), C)^* = \text{Hom}_{C^*}(\text{Hom}_{C^*}(R\Gamma(X^*), C), C) \cong R\Gamma(X^*)$$

since ${}_C C_{C^*}$ defines a Morita duality. Hence we have natural isomorphisms in $\mathcal{D}_{qf}^b(\mathcal{M}^C)$

$$R\Gamma(\text{Hom}_{C^*}(R\Gamma(X^*), C)^*) \cong R\Gamma(R\Gamma(X^*)) \cong R\Gamma(X^*),$$

where the last isomorphism holds because $R\Gamma(X^*)$ is a complex of J -torsion modules as an object in $\mathcal{D}_{fg}^b({}^C\mathcal{M})$. So, Ψ and F are naturally isomorphic. Similarly, we see G is naturally isomorphic to the following composite functor:

$$\Psi : \mathcal{D}_{qf}^b(\mathcal{M}^C) \xrightarrow{(\quad)^*} \mathcal{D}_{fg}^b(\mathcal{M}_{C^*}) \xrightarrow{\text{Hom}_{C^*}(R\Gamma^\circ(-), C)} \mathcal{D}_{fg}^b({}^C\mathcal{M}) \xrightarrow{R\Gamma^\circ \circ (\quad)^*} \mathcal{D}_{qf}^b({}^C\mathcal{M}).$$

Then we get the desired results. \square

Remark 2.6. *When C is not artinian, there may exist certain dualities between the derived categories of comodules. For example, let C be a (both left and right) semiperfect coalgebra, that is, the categories ${}^C\mathcal{M}$ and \mathcal{M}^C have enough projective objects (cf. [17]). Then the rational functor Rat is exact [12, 13, 18], and $\text{Rat} \circ (\quad)^*$ gives a Colby-Fuller duality [12, Theorem 3.5]:*

$$\mathcal{M}^C \begin{array}{c} \xrightarrow{\text{Rat} \circ (\quad)^*} \\ \xleftarrow{\text{Rat} \circ (\quad)^*} \end{array} {}^C\mathcal{M}.$$

Of course, this duality induces a duality between certain triangulated subcategories of the derived categories of \mathcal{M}^C and ${}^C\mathcal{M}$ respectively.

There are two questions arising from the above theorem: (i) when does an artinian coalgebra satisfy the χ -condition? (ii) when is $F({}^C C)$ isomorphic to ${}^C C$ in $\mathcal{D}_{qf}^b(\mathcal{M}^C)$? We deal with these questions in the rest of the paper.

3. COALGEBRAS SATISFY THE χ -CONDITION

In this section, we introduce a class of coalgebras dual to Artin-Schelter algebra. These coalgebras satisfy the χ -condition of Theorem 2.5.

The classical concept of Artin-Schelter (AS, for short) regular algebra [3] is defined over graded algebras. We may extend this concept from graded algebras to semiperfect algebras. We say that a noetherian semiperfect algebra A is *left AS-regular* (cf. [9]) if A has finite global dimension d and for every left simple A -module S , one has

$$\text{Ext}_A^i(S, A) = \begin{cases} 0, & \text{if } i \neq d; \\ T, & \text{if } i = d, \end{cases}$$

where T is a right simple A -module. Similarly, one can define a *right AS-regular* algebra.

In the dual case, we have the following definition.

Definition 3.1. Let C be an artinian coalgebra with global dimension $d < \infty$. We say that C is *right AS-regular* if, for every simple left C^* -module S , we have

$$\text{Ext}_{C^*}^i(C, S) = \begin{cases} 0, & \text{if } i \neq d; \\ T, & \text{if } i = d, \end{cases}$$

where T is a simple left C^* -module.

Similarly, we may define a *left AS-regular coalgebra*.

Remark 3.2. *In the definition above, we may use alternatively the extensions in the category of comodules to define an AS-regular coalgebra. In fact, we have $\text{Ext}_C^i(C, S) = \text{Ext}_{C^*}^i(C, S)$ since the subcategory ${}_{C^*}\text{Rat}(\mathcal{M})$ is a thick subcategory. Also $T = \text{Ext}_C^d(C, S)$ in the definition above is a right C -comodule.*

Following Prop. 1.2, one can check without difficulty that the following holds.

Proposition 3.3. *A coalgebra C is left AS-regular if and only if C^* is left AS-regular.*

A graded AS-regular algebra must satisfy the χ -condition (cf. [3], Sec.8). Similarly, a nongraded AS-regular algebra also satisfies the χ -condition (cf. [9]). In the dual case, we have the following.

Proposition 3.4. *If C is a left AS-regular coalgebra, then C satisfies the left χ -condition.*

Proof. Note that the AS-regularity of C and that of C^* are equivalent. □

We will see that the concept of an AS-regular algebra (coalgebra) is left-right symmetric. Let A be a left AS-regular algebra of global dimension d . If ${}_A S$ is a simple module, then we write S^\natural for the right simple module $\text{Ext}_A^d(S, A)$.

Lemma 3.5. *Let A be a left AS-regular algebra of global dimension d . Then for two left simple A -modules S and T , $S^\natural \cong T^\natural$ if and only if $S \cong T$.*

Proof. Let

$$0 \longrightarrow P^{-d} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow S \longrightarrow 0,$$

and

$$0 \longrightarrow Q^{-d} \longrightarrow \cdots \longrightarrow Q^{-1} \longrightarrow Q^0 \longrightarrow T \longrightarrow 0$$

be minimal projective resolutions of S and T respectively. Applying the functor $\text{Hom}_A(-, A)$ to both resolutions, we obtain exact sequences:

$$(3) \quad 0 \longrightarrow \text{Hom}_A(P^0, A) \longrightarrow \text{Hom}_A(P^{-1}, A) \longrightarrow \cdots \longrightarrow \text{Hom}_A(P^{-d}, A) \longrightarrow S^\natural \longrightarrow 0,$$

and

$$(4) \quad 0 \longrightarrow \text{Hom}_A(Q^0, A) \longrightarrow \text{Hom}_A(Q^{-1}, A) \longrightarrow \cdots \longrightarrow \text{Hom}_A(Q^{-d}, A) \longrightarrow T^\natural \longrightarrow 0.$$

Since P^i and Q^i are finitely generated projective modules for all $i \geq 0$, the above sequences are minimal projective resolutions of S^\natural and T^\natural respectively. Suppose $S^\natural \cong T^\natural$. Then the sequences (3) and (4) are isomorphic. Notice that we have isomorphisms of complexes

$$\text{Hom}_A(\text{Hom}_A(P^\bullet, A), A) \cong P^\bullet$$

and

$$\mathrm{Hom}_A(\mathrm{Hom}_A(Q^\bullet, A), A) \cong Q^\bullet.$$

Thus we obtain $S \cong T$. \square

Proposition 3.6. *A noetherian semiperfect algebra is left AS-regular if and only if it is right AS-regular.*

As a consequence, an artinian coalgebra is left AS-regular if and only if it is right AS-regular.

Proof. Suppose that A is left AS-regular. Let ${}_A S$ be a simple module. By the proof of Lemma 3.5, the right simple module S^\natural has a minimal projective resolution (3), and $\mathrm{Ext}_A^i(S^\natural, A) = 0$ for $i \neq d$ and $\mathrm{Ext}_A^d(S^\natural, A) = S$. Since there are only finitely many nonisomorphic simple left (and right) A -modules, by Lemma 3.5, for every right simple A -module K there is a left simple A -module S such that $K = S^\natural$. Hence A is right AS-regular. \square

In view of the proposition above, we may omit the prefix “left” and “right” and just say an AS-regular (co)algebra.

Let C be an artinian coalgebra satisfying left and right χ -conditions. The local cohomology of C^* (relative to the Jacobson radical) provides a duality between certain triangulated categories (see Theorem 2.5). Furthermore, it gives a ‘balanced’ dualizing complex of C^* (cf. [9], for the terminology). If C is an AS-regular coalgebra, we can compute the local cohomology of C^* . Recall that a coalgebra is *basic* if the dual of any simple subcoalgebra is a division algebra (cf. [6]). If C is a basic artinian coalgebra, then C^* is a noetherian basic algebra.

Theorem 3.7. *If C is a basic AS-regular coalgebra, then there is a coalgebra automorphism $\sigma \in \mathrm{Aut}(C)$ and a nonnegative integer n such that $R\Gamma(C^*) \cong {}_1C_{\sigma^*}[-n]$ in $\mathcal{D}^b({}_1C^*\mathcal{M}_{C^*})$.*

Proof. When C^* is a local algebra, the result can be deduced from [9, Cor. 3.9]. But it seems that we could not extend the proof in [9] directly to the general case.

Since we work over an algebraically closed field, any simple comodule over the basic coalgebra C is one-dimensional. Assume C is of global dimension n . Let

$$(5) \quad 0 \longrightarrow {}_{C^*}C^* \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow 0$$

be the minimal injective resolution of the left module ${}_{C^*}C^*$. Let ${}_{C^*}S$ be any simple C^* -module. We have $\mathrm{Ext}_{C^*}^i(S, C^*) \cong \mathrm{Hom}_{C^*}(S, I^i)$ for all $i \geq 0$. By the AS-regularity of C , we obtain $\mathrm{Hom}_{C^*}(S, I^i) = 0$ for $i \neq n$ and $\dim \mathrm{Hom}_{C^*}(S, I^n) = 1$. Then we have $\mathrm{Soc}(I^i) = 0$ for all $i < n$ and $\mathrm{Soc}(I^n) = {}_{C^*}C_0$, where C_0 is the coradical of C . Hence I^i is J -torsion free for all $i < n$, and $I^n = {}_{C^*}C \oplus \bar{I}^n$ for some J -torsion free

module \bar{I}^n . Therefore we get $R\Gamma(C^*) \cong {}_{C^*}C[-n]$ in $\mathcal{D}^b({}_{C^*}\mathcal{M})$. Note that $R\Gamma(C^*)$ is an object in $\mathcal{D}^b({}_{C^*}\mathcal{M}_{C^*})$. Let $T = H^{-n}(R\Gamma(C^*)^*)$. Then T is a C^* -bimodule, and T is free as a right C^* -module. Since C (and dually C^*) satisfies left and right χ -conditions, by [22, Theorem 4.1], T is a dualizing complex of C^* . Since C^* is of finite global dimension, C^* itself is a dualizing complex. By [24, Theorem 4.5], T is a tilting complex. Now applying the right version of [20, Prop. 2.3], we obtain an algebra automorphism $\alpha \in \text{Aut}(C^*)$ such that $T \cong {}_{\alpha}C_1^*$. Since C is reflexive, there is a unique coalgebra automorphism $\sigma \in \text{Aut}(C)$ such that $\sigma^* = \alpha$. Let $U = H^n(R\Gamma(C^*))$. It is clear that $T = U^*$. Since $U \cong {}_{C^*}C$ as a left C^* -module, $U^* \cong \text{Hom}_{C^*}(U, C)$ as C^* -bimodules. Since ${}_{C^*}C_{C^*}$ defines a Morita duality, the canonical morphism $U \rightarrow \text{Hom}_{C^* \circ p}(\text{Hom}_{C^*}(U, C), C)$ is an isomorphism of C^* -bimodules. Thus we have $U \cong \text{Hom}_{C^* \circ p}(T, C) \cong {}_1C_{\alpha} = {}_1C_{\sigma^*}$ as C^* -bimodules. Therefore $R\Gamma(C^*) \cong {}_1C_{\sigma^*}[-n]$. \square

Remark 3.8. (i) We call the automorphism $\sigma \in \text{Aut}(C)$ the Nakayama automorphism of C , and call its dual automorphism σ^* the Nakayama automorphism of C^* . Note that Nakayama automorphism is unique up to inner automorphisms.

(ii) The theorem implies that ${}_{\sigma^*}C_1^*[n]$ is the balanced dualizing complex (cf. [9]) of C^* .

(iii) Let A be a noetherian complete (with respect to the Jacobson radical) basic algebra, and let $C(:= A^{\circ})$ be its dual coalgebra. Then C is an artinian basic coalgebra and $A = C^*$ (cf. [15, Prop. 4.3.1]). So, the preceding theorem applies to all the noetherian complete AS-regular basic algebra.

From Theorem 3.7, one can deduce the following finiteness properties of extension groups of finitely generated C^* -modules, which can be viewed as a generalization of [2, Prop. 2.46(ii,iii)] and [25, Theorem 0.3(4)].

Corollary 3.9. Let C be a basic AS-regular coalgebra of global dimension n . If M is a finitely generated left (or right) C^* -module, then:

- (i) $\dim \text{Ext}_{C^*}^n(M, C^*) < \infty$; moreover, as vector spaces $\text{Ext}_{C^*}^n(M, C^*) \cong \text{Rat}(M)^*$;
- (ii) for $i < n$, $\text{Ext}_{C^*}^i(M, C^*) \cong \text{Ext}_{C^*}^i(M/\text{Rat}(M), C^*)$.

Proof. (i) By Theorem 3.7, $R\Gamma(C^*) \cong {}_{\sigma^*}C_1^*[n]$ in $\mathcal{D}^b({}_{C^*}\mathcal{M}_{C^*})$. Applying the local duality theorem [9, Prop 3.4], we have

$$R\Gamma(M)^* \cong \text{RHom}_{C^*}(M, {}_{\sigma^*}C_1^*[n]).$$

Taking the 0-th cohomology on both sides of the complexes above, we obtain

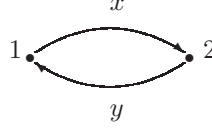
$$\text{Rat}(M)^* \cong \text{Hom}_{\mathcal{D}^b({}_{C^*}\mathcal{M})}(M, {}_{\sigma^*}C_1^*[n]) \cong \text{Ext}_{C^*}^n(M, {}_{\sigma^*}C_1^*).$$

Since M is finitely generated, $\text{Rat}(M)$ is finite dimensional.

(ii) From the proof of the preceding theorem, in the minimal injective resolution (5) of C^* , I^i is torsion free for all $i < n$. Then we have $\text{Hom}_{C^*}(M, I^i) \cong \text{Hom}_{C^*}(M/\text{Rat}(M), I^i)$. Hence (ii) follows. \square

We end this section with an example of basic artinian AS-regular coalgebra.

Example 3.10. Let Q be the following quiver:



Let C be the path coalgebra CQ . Let S_1 and S_2 be the simple left C -comodules corresponding to the vertices, and $e_1, e_2 \in C^*$ be the idempotents corresponding to the vertices. Now the injective envelop of S_1 is e_1C , and the injective envelop of S_2 is e_2C . One easily sees that any quotient comodule of e_1C (or e_2C) is left quasi-finite. Hence C is strictly quasi-finite as a left C -comodule (cf. [14, Theorem 3.1]). Similarly, C is also strictly quasi-finite as a right C -comodule. Hence C is artinian. The minimal injective resolution of S_1 is

$$0 \longrightarrow S_1 \longrightarrow e_1C \xrightarrow{f} e_2C \longrightarrow 0,$$

where $f(x) = y^* \cdot x$ for all $x \in e_1C$, where $y^* \in C^*$ is the linear map sending y to the unit and other paths to 0. We want to compute the kernel and cokernel of the map

$\text{Hom}_C(C, e_1C) \xrightarrow{\text{Hom}_C(C, f)} \text{Hom}_C(C, e_2C)$. Let $h_C(-, -)$ be the cohom functor. We have the commutative diagram of morphisms of left C^* -modules (cf. [16, Appendix]):

$$\begin{array}{ccc} h_C(e_1C, C)^* & \xrightarrow{h_C(f, C)^*} & h_C(e_2C, C)^* \\ \downarrow \cong & & \downarrow \cong \\ \text{Hom}_C(C, e_1C) & \xrightarrow{\text{Hom}_C(C, f)} & \text{Hom}_C(C, e_2C). \end{array}$$

By [16, Appendix], we have a commutative diagram of morphisms of right C -comodules:

$$\begin{array}{ccc} h_C(e_2C, C) & \xrightarrow{h_C(y^* \cdot, C)} & h_C(e_1C, C) \\ \downarrow \cong & & \downarrow \cong \\ Ce_2 & \xrightarrow{\cdot y^*} & Ce_1. \end{array}$$

It is clear that the bottom map is surjective and the kernel of the bottom map is S_2 . Let T_1 and T_2 be the simple right C -comodules corresponding to the vertices. Then, $\text{Ext}_C^0(C, S_1) = 0$ and $\text{Ext}_C^1(C, S_1) \cong T_2$ as right C -comodules.

Similarly, we see $\text{Ext}_C^0(C, S_2) = 0$ and $\text{Ext}_C^1(C, S_2) \cong T_1$. Hence C is an AS-regular coalgebra, and C^* is an AS-regular algebra.

4. CALABI-YAU PROPERTY OF AS-REGULAR (CO)ALGEBRAS

In this section we give the relations between the CY property and AS-regularity of noetherian complete semiperfect algebras (or equivalently, artinian coalgebras).

Let \mathcal{C} be a Hom-finite \mathbf{k} -linear category. Recall that an additive functor $\mathcal{S} : \mathcal{C} \rightarrow \mathcal{C}$ is called a *right Serre functor* (cf. [4, Appendix]) if there are natural isomorphisms

$$\eta_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(Y, \mathcal{S}X),$$

for any $X, Y \in \mathcal{C}$. Note that a Hom-finite \mathbf{k} -linear triangulated category is CY- n if and only if the n -th shift functor $[n]$ is a right Serre functor.

Let A be a noetherian algebra. Then the triangulated category $\mathcal{D}_{fd}^b({}_A\mathcal{M})$ is a Hom-finite \mathbf{k} -linear category. If A satisfies the left χ -condition and ${}_AA$ has finite injective dimension, then $\mathcal{D}_{fd}^b({}_A\mathcal{M})$ has a Serre functor.

Lemma 4.1. *Let A be a noetherian semiperfect algebra with cofinite Jacobson radical. If A satisfies the left χ -condition and ${}_AA$ has finite injective dimension, then $\mathcal{S} = \text{RHom}_A(-, A)^*$ is a right Serre functor of $\mathcal{D}_{fd}^b({}_A\mathcal{M})$.*

Proof. We only point out that the χ -condition ensures that \mathcal{S} is a well defined functor from $\mathcal{D}_{fd}^b({}_A\mathcal{M})$ to itself. \square

Let C be an artinian coalgebra satisfying the left χ -condition. In this case, we have a natural isomorphism $\text{Hom}_{C^*}(X, C) \cong X^*$ for every $X \in \mathcal{D}_{fd}^b(C^*\mathcal{M})$ in $\mathcal{D}_{fd}^b(\mathcal{M}_{C^*})$. Indeed, by [22, Lemma 2.6], we have $X \cong R\Gamma(X)$ in $\mathcal{D}_{fd}^b(C^*\mathcal{M})$. Since $R\Gamma(X)$ is also an object in $\mathcal{D}_{fd}^b(\mathcal{M}^C)$, we have $\text{Hom}_{C^*}(R\Gamma(X), C) \cong R\Gamma(X)^*$.

Theorem 4.2. *Let C be a basic AS-regular coalgebra of global dimension n , and let σ be the Nakayama automorphism of C . Then there are natural isomorphisms*

$$\mathcal{S}(X) \cong \sigma^* X[n],$$

for all $X \in \mathcal{D}_{fd}^b(C^*\mathcal{M})$.

Proof. Set $A := C^*$ and $\alpha := \sigma^*$. For $X \in \mathcal{D}_{fd}^b({}_A\mathcal{M})$, we have natural isomorphisms in $\mathcal{D}_{fd}^b(\mathcal{M}_A)$:

$$\begin{aligned} \mathcal{S}(X)^* &= \text{RHom}_A(X, A) \\ &\cong \text{RHom}_A(X, {}_1A_{\alpha^{-1}}[n]) \otimes_A {}_1A_{\alpha}[-n] \\ &\stackrel{(a)}{\cong} \text{Hom}_A(R\Gamma(X), C) \otimes_A {}_1A_{\alpha}[-n] \\ &\cong R\Gamma(X)^* \otimes_A {}_1A_{\alpha}[-n] \\ &\cong X^* \otimes_A {}_1A_{\alpha}[-n]. \end{aligned}$$

The isomorphism (a) above follows from the local duality theorem [9, Prop. 3.4] since ${}_1A_{\alpha^{-1}}[n]$ is the balanced dualizing complex of A (cf. Remark 3.8). Now by

taking the \mathbf{k} -linear dual of the above isomorphisms, we obtain a natural isomorphism $\mathcal{S}(X) \cong {}_{\alpha}X[n]$. \square

Recall that an automorphism of C is *inner* if its dual is an inner automorphism of C^* . Clearly, if the Nakayama automorphism σ is inner, then $R\Gamma(C^*) \cong C[-n]$. The converse is also true. So, immediately, we obtain the following criterion for the dual algebra of an AS-regular coalgebra to be CY.

Corollary 4.3. *Let C be a basic AS-regular coalgebra. If the Nakayama automorphism of C is inner, then C^* is CY.*

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